Int. J. Heat Mass Transfer. Vol. 10, pp. 1327–1338. Pergamon Press Ltd. 1967. Printed in Great Britain

A STUDY OF VARIOUS LIMITS IN RADIATION HEAT-TRANSFER PROBLEMS

L. S. WANG and C. L. TIEN

State University of New York at Stony Brook, and University of California, Berkeley

(*Received* 10 *September* 1966)

Abstract—A study is presented on the two classes of limiting analyses in radiation heat-transfer problems: the analyses of the opaque limit and the transparent limit on the one hand and those of the radiation predominant limit and the conduction predominant limit on the other hand. In the first case treatment valid uniformly regardless whether radiation predominant or conduction predominant is not available. In the second case, however, a uniformly valid treatment regardless of the value of opacity is achieved. Accordingly, it offers a more logical way than the first case to the solution of combined radiation and conduction problems. The scheme comprises the solution of a non-linear differential equation in the radiation predominant case and a non-linear integral equation in the conduction predominant case.

NOMENCLATURE

- C, parameter which determines the relative importance of conduction and radiation on the determination of temperature profile;
- E_n, exponential integral of nth order;
- G, emerging intensity from the medium across the boundary;
- H, unit step function;
- I', intensity;
- I, incident intensity across the boundary on the medium;
- J, mean intensity, $J \equiv \frac{1}{2} \int_{-1}^{+1} I' \mathrm{d}\mu$;
- J_0, mean intensity of a non-absorbing medium;
- J_r, mean intensity of a medium in radiative equilibrium;
- k, thermal conductivity;
- L, thickness of a plane layer;
- N, conduction–radiation interaction parameter defined by $N \equiv k\rho\alpha/4\sigma T_1^3$;
- q, heat flux;
- S, source function;
- T, temperature;
- $\tilde{T}$, equivalent temperature of a diffusely incident intensity;
- x, coordinate perpendicular to boundary.

Greek symbols

- α, absorption coefficient;
- δ, Dirac delta function;
- ϵ, emissivity of an opaque wall;
- θ, dimensionless temperature, $\theta = T/T_1$;
- μ, directional cosine between the intensity and the outward drawn normal of the boundary;
- ξ, dimensionless coordinate, $\xi \equiv x/L$;
- ρ, density;
- τ, optical depth;
- τ_0, optical thickness of a plane layer, $\tau_0 \equiv \rho\alpha L$.

Subscripts

- c, refers to conduction;
- r, refers to radiation;
- 1, 2, refer to boundary 1 and boundary 2 respectively.

Superscript

- $+$, denotes dimensionless quantity.

1. INTRODUCTION

THERE are two special cases for which the radiation flux can be expressed in simple formulas [1, 2]: (i) opaque materials, in which the absorption of radiation is so intense that radiation is in local thermodynamic equilibrium* with the matter, and (ii) transparent materials, in which there is little self-absorption of the radiated energy. These two limits have been successfully applied [3, 4] to the calculation of radiation heat flux through mediums in the absence of conduction.

The application of these two limits to the calculation of heat flux through mediums in the presence of both radiation and conduction was given by Cess [2] with limited success. Wang and Tien [5], instead of considering the opaque and transparent limits, obtained satisfactory results by considering the radiation predominant and conduction predominant limits.

In view of the existence of two sets of limiting situations it seems advantageous to discuss them in one single paper in order to examine their respective limitations and to see their relationship. This paper is composed of four parts. In Section 2 the opaque and transparent limits of radiation heat transfer in the absence of conduction are briefly reviewed. The contribution there is the clarification of the boundary conditions. In Section 3 a treatment of the opaque and transparent limits of combined radiation and conduction is presented. The treatment has limited applicability because of the difficulty in the imposing of boundary conditions. Presented in Section 4 is a treatment of the radiation predominant and conduction predominant limits. The contribution there is the development of a formulation which is not based on the approximate differential method as was used by Wang and Tien [5] in a previous paper. In Section 5 the respective limitations of both approximate methods are then discussed together.

Throughout this paper only thermal radiation,† i.e. the radiation source function depends only on temperature, not on radiation mean intensity, will be considered. And only finite, one-dimensional systems will be studied.

* The meaning of "local thermodynamic equilibrium" throughout this paper is to be understood in the conventional usage in thermodynamics literature while "thermal radiation" will be used to substitute the often used "local thermodynamic equilibrium" in radiation literature.

† See previous footnote.

2. RADIATIVE EQUILIBRIUM

2.1 *The opaque limit*

The radiation flux in the opaque limit is equal to [6]

$$\vec{q}_r = -\frac{4\pi}{3\rho\alpha}\operatorname{grad} J, \tag{1}$$

where $\vec{q}_r$ is the radiation flux, $\rho\alpha$ the extinction coefficient which is proportional to density, ρ, and J the mean intensity. Integration of the equation of transfer with respect to all directions at a given point yields

$$\operatorname{div}\vec{q}_r = 4\pi\rho\alpha(S - J), \tag{2}$$

where S is the source function which is assumed to be equal to the Planck function in this study. In radiative equilibrium,

$$\operatorname{div}\vec{q}_r = 0, \tag{3}$$

hence

$$S = J. \tag{4}$$

Since conduction is considered to be negligible in this section, the temperature itself at the boundary is not a boundary condition to the medium. Precisely speaking, the boundary condition is the "continuity of the radiation heat flux and of the mean intensity at the boundary". Suppose the incident intensity across the boundary on the medium is $I(\mu)$, where μ is the directional cosine between the incident intensity and the outward drawn normal of the boundary. And suppose the emerging intensity from the medium across the boundary is $G(\mu)$. Then the radiation heat flux at the boundary is equal to

$$q_r = 2\pi\int_0^1 \mu G(\mu)\,\mathrm{d}\mu + 2\pi\int_{-1}^0 \mu I(\mu)\,\mathrm{d}\mu. \tag{5}$$

As the medium is opaque,

$$G(\mu) \cong G = \text{constant};$$

hence

$$q_r = \pi G + 2\pi \int_{-1}^{0} \mu I(\mu)\, \mathrm{d}\mu. \qquad (6)$$

And the mean intensity at the boundary is equal to

$$J = \tfrac{1}{2}\int_{0}^{1} G(\mu)\, \mathrm{d}\mu + \tfrac{1}{2}\int_{-1}^{0} I(\mu)\, \mathrm{d}\mu$$

$$\cong \tfrac{1}{2}G + \tfrac{1}{2}\int_{-1}^{0} I(\mu)\, \mathrm{d}\mu. \qquad (7)$$

When $I(\mu)$ is explicitly given, equations (6) and (7) become the boundary condition of equation (1) after eliminating G from them. This is the case of a transparent wall boundary.

However, in the case that the boundary is an opaque wall the temperature of the wall, not the incident intensity, is the condition explicitly given. The following formula,

$$q_r = \frac{\epsilon}{1-\epsilon}(\pi I - \sigma T^4), \qquad (8)$$

where ϵ is the emissivity of the wall, is then needed to determine the incident intensity, I, from the known wall temperature and wall emissivity. The opaque wall is assumed to be diffuse in deriving equation (8).

To illustrate the applicability of the above formulation to various boundary conditions a plane layer of thickness, L, with black opaque wall at temperature T_1, on one side is considered. The radiation heat flux results are:

(i)

$$q_r = \frac{\sigma T_1^4 - \sigma T_2^4}{(1/\epsilon_2 - 1) + (1 + \frac{3}{4}\rho\alpha L)}, \qquad (9)$$

when the other boundary is an opaque wall with emissivity ϵ_2 and at temperature T_2;

(ii)

$$q_r = \frac{\sigma T_1^4 - \sigma \tilde{T}^4}{1 + \frac{3}{4}\rho\alpha L}, \qquad (10)$$

when the other boundary is a transparent wall which separates the medium in the plane layer from an isothermal enclosure at temperature $\tilde{T}$;

and

(iii)

$$q_r = \frac{\sigma T_1^4 - \frac{1 + 2|\mu_0|}{2}\pi I_2}{1 + \frac{3}{4}\rho\alpha L}, \qquad (11)$$

when the other boundary is a transparent wall and a parallel radiation beam of intensity I_2 is incident on the plane layer with an angle $\theta_0 = \cos^{-1}\mu_0$, i.e.

$$I(\mu) = I_2\delta(\mu - \mu_0). \qquad (12)$$

Equations (10) and (11) indicate that a transparent wall is equivalent to a black opaque wall at some equivalent temperature, for case (ii)

$$T = \tilde{T} \qquad (13a)$$

and for case (iii)

$$T = \left(\frac{1 + 2|\mu_0|}{2\sigma}\pi I_2\right)^{\frac{1}{4}}. \qquad (13b)$$

The present results agree asymptotically with the existing rigorous results [7].

With boundary conditions properly chosen, the differential approximation, equation (1), is seen to be quite flexible in its application. The temperature slip determined by equation (4) is an inherent feature of a medium in radiative equilibrium when it is adjoining an opaque wall. It is the mean intensity, not the temperature, which satisfies the non-slip condition.

2.2 *The transparent limit*

In the transparent limit, the mean intensity can be written as

$$J(\rho\alpha, \vec{x}) = J(0, \vec{x}) + \rho\alpha \frac{\mathrm{d}}{\mathrm{d}(\rho\alpha)} J(0, \vec{x}) + \ldots. \qquad (14)$$

As a first approximation in the transparent limit, therefore, the source function and the mean intensity may be taken as

$$S = J \cong J(0, \vec{x}) \equiv J_0(\vec{x}). \qquad (14a)$$

Here J_0 is the mean intensity of a non-absorbing medium which is a function of position only. Now at a certain point on the boundary 1 the radiation heat flux, given by

$$q_r = 2\pi\int_0^1 \mu G(\mu)\,\mathrm{d}\mu + 2\pi\int_{-1}^0 \mu I(\mu)\,\mathrm{d}\mu, \quad (15)$$

is approximately equal to

$$q_r \cong \pi\rho\alpha\overline{J_0 L} + \sum_i [1 - (\pi/w_i)\,\rho\alpha\overline{L_i}]\,w_i I_i - \pi I_1, \quad (15a)$$

where $\overline{L}$ is the geometric mean beam length of the medium volume to the point of interest, $\overline{L_i}$ the geometric mean beam length of the conical volume subtended by the ith boundary surface to the point of interest,

$$\sum_i \overline{L_i} = \overline{L},$$

w_i the projection area, from the surface segment of a hemisphere of unit radius subtended by the ith boundary surface, on the base plane which is tangent to boundary surface 1 at the point of interest,

$$\sum_i w_i = \pi,$$

and $\overline{J_0 L}$ the integral of J_0 with respect to $\overline{L}$,

$$\overline{J_0 L} \equiv \int_{\overline{L}} J_0(T)\frac{\mathrm{d}\overline{L}}{\mathrm{d}T}\,\mathrm{d}T,$$

the detail of which is referred to [8]. Equation (15a) is only valid for diffusely incident radiation intensity. The second and third terms on the right hand side of equation (15a) may, of course, be replaced by appropriate terms as was done in Section 2.1 when I depends on directions.

As an example of solving the transparent limit problems let us consider a plane layer with opaque walls of emissivities ϵ_1 and ϵ_2 on boundary 1 and boundary 2 respectively. Using equations (6) and (8) and setting $G_1 = I_2$ and $G_2 = I_1$ yield the transparent mean intensity

$$J_0 = \frac{\left(\frac{1}{\epsilon_2} - \frac{1}{2}\right)\frac{\sigma}{\pi}T_1^4 + \left(\frac{1}{\epsilon_1} - \frac{1}{2}\right)\frac{\sigma}{\pi}T_2^4}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}. \quad (16)$$

For a plane layer $\overline{L} = 2L$; and $\overline{J_0 L} = J_0\overline{L}$ because J_0 is constant. Equation (15a) then becomes

$$q_r = 2\pi(\rho\alpha L)\,J_0 + (1 - 2\rho\alpha L)\,\pi I_1 - \pi I_2 \quad (17a)$$

and

$$q_r = -\,2\pi(\rho\alpha L)\,J_0 - (1 - 2\rho\alpha L)\,\pi I_2 + \pi I_1, \quad (17b)$$

and consequently

$$q_r = (\pi I_1 - \pi I_2)(1 - \rho\alpha L). \quad (17)$$

This may be rewritten as

$$q_r = \frac{\pi I_1 - \pi I_2}{1 + \rho\alpha L}, \quad (18)$$

which is consistent with the transparent approximation made and avoids yielding the physically impossible results as equation (17) does when $\rho\alpha L > 1$. The present results, equation (17) or equation (18), also agree asymptotically with the existing rigorous results [7].

3. COMBINED RADIATION AND CONDUCTION—PART A

Under the influence of conduction the finite difference between the temperature of an opaque wall and that of the medium touching the opaque wall disappears. (Discontinuity of temperatures at the walls is of course also associated with conduction, but it is not significant in a system many conduction-mean-free-paths thick. The ratio $l_{\mathrm{rad}}/l_{\mathrm{cond}}$ is generally a number of many orders of magnitude.) The disappearance of temperature slip does not imply, however, that the boundary condition of the continuity of the radiation heat flux and of the mean intensity is replaced by that of the continuity of temperature. Rather, it introduces a new boundary condition —the continuity of temperature—in addition to the radiation boundary condition. This should be clear for the case where the boundary is a transparent wall which separates the medium from an isothermal enclosure. As far as the radiation incident intensity is concerned, then, this boundary is equivalent to a black opaque wall at the temperature of the isothermal enclosure. But the medium temperature at the boundary must also be equal to the temperature of the transparent wall which may be controlled independently. Therefore for this case there are two independently prescribed temperatures in the boundary conditions. For the case where the boundary is an opaque wall, both boundary conditions still have to be satisfied. There exists, however, a relationship between them because they are both described by the same temperature.

3.1 *The opaque limit*

In the opaque limit the radiation heat flux far away from boundary is equal to

$$\vec{q}_r = -\frac{4\pi}{3\rho\alpha}\operatorname{grad} J \cong -\frac{4\pi}{3\rho\alpha}\operatorname{grad} S. \quad (19)$$

The second approximate equality is a consequence of equation (2).

It is well known that the approximation represented by equation (19) becomes less satisfactory in the regions near the boundary. A recent paper [9] is devoted to obtaining a uniformly valid approximation in the opaque limit. A simpler treatment without going through the unnecessary complications in [9] is presented in this section.

Since the boundary effect is confined within a geometrically thin layer, it is sufficient to consider a plane boundary layer when the radius of curvature of the boundary surface is not too small. Suppose that the effect of conduction is so significant that the temperature is an analytic function everywhere, then the source function (or the Planck function), being an analytic function of temperature, must also be an analytic function and can be expanded into a Taylor's series. Define the optical depth, $\tau \equiv \rho\alpha x$, where x is the normal distance from the boundary surface. The equation of transfer is then integrated with respect to τ to give

$$I' = I\,\mathrm{e}^{\tau/\mu} - \mathrm{e}^{\tau/\mu}\int_0^\tau S\,\mathrm{e}^{-t/\mu}\frac{\mathrm{d}t}{\mu}, \quad (20)$$

where I' is the intensity. The Taylor's expansion of the source function is

$$S(t) = S(\tau) + (t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}S(\tau) + \tfrac{1}{2}(t-\tau)^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}S(\tau) + \ldots. \quad (21)$$

Substituting equation (21) into the integrand in equation (20) and carrying out the integration yield

$$I' = I\,\mathrm{e}^{\tau/\mu} - \mathrm{e}^{\tau/\mu} \times \left[S(0) + \mu\frac{\mathrm{d}}{\mathrm{d}\tau}S(0) + \mu^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}S(0) + \right]\ldots + \left[S(\tau) + \mu\frac{\mathrm{d}}{\mathrm{d}\tau}S(\tau) + \mu^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}S(\tau) + \ldots\right]. \quad (21a)$$

The radiation heat flux then becomes after integration equation (21a) with respect to μ

$$q_r(\tau) = \left[2\pi(I - S(0))E_3(\tau) + 2\pi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}S(0)\right)E_4(\tau)\right] - \frac{4\pi}{3}\frac{\mathrm{d}}{\mathrm{d}\tau}S(\tau). \quad (22)$$

The first part on the right-hand side of equation (22) represents the boundary effect which decays exponentially with the optical depth. When I and $S(0)$ are given independently, the substitution of equation (22) in its present form into the energy conservation equation would yield the solution to the problem. For diffuse, opaque walls, however, I and $S(0)$ are related by

$$\pi I = \epsilon\pi S(0) + (1-\epsilon)\int_0^1 2\pi\mu\,\mathrm{d}\mu\int_0^{\tau_0} S(t)\,\mathrm{e}^{-t/\mu}\frac{\mathrm{d}t}{\mu}$$

$$\cong \pi S(0) + \frac{2\pi}{3}(1-\epsilon)\frac{\mathrm{d}}{\mathrm{d}\tau}S(0). \quad (23)$$

The radiation heat flux becomes at the boundary

$$q_r = -\frac{2\pi}{3}\frac{\mathrm{d}}{\mathrm{d}\tau}S(0) + \pi(I - S(0)), \quad (24a)$$

or, for opaque walls,

$$q_r = -\frac{2\pi}{3}\epsilon\frac{d}{d\tau}S(0). \tag{24b}$$

By comparing equation (24b) with equation (19) a difference of factor $\epsilon/2$ is seen between the heat flux at the boundary and the heat flux far away from the boundary. Therefore the temperature gradient is generally steeper near the boundary.

Consider a plane layer again for example. Substituting equation (22) into the equation of conservation of energy and using the temperature continuity boundary condition result in the total heat flux in terms of the driving potentials at both boundaries

$$\frac{3\tau_0}{4}q^+ = 3N(1-\theta_2) + (1-\theta_2^4)$$
$$\tfrac{1}{2}(I_1^+ - 1) + \tfrac{1}{2}(\theta_2^4 - I_2^+)$$
$$+\frac{3}{2}\frac{d}{d\tau}\theta(0) + \tfrac{3}{2}\theta_2^3\frac{d}{d\tau}\theta(\tau_0), \tag{25}$$

where

$$\tau_0 = \rho\alpha L, \qquad N = \frac{k\rho\alpha}{4\sigma T_1^3},$$

$$\theta_2 = T_2/T_1, \qquad \theta = T/T_1, \qquad q^+ = \frac{q}{\sigma T_1^4},$$

$$I_1^+ = \frac{I_1}{(\sigma/\pi)T_1^4}, \qquad I_2^+ = \frac{I_2}{(\sigma/\pi)T_1^4}.$$

The total heat flux given by equation (25) must equal the total heat fluxes at both boundaries obtained by using equation (24) for the radiation heat flux and Fourier's law for the conduction heat flux. Eliminating $d\theta(0)/d\tau$ and $d\theta(\tau_0)/d\tau$ among them then yields

The above analysis reveals that for opaque walls the modified Rosseland approximation, equation (22), leads to a correction of order $(1/\tau_0)$ on the total heat flux. Hence, the Rosseland approximation, equation (19), is as valid in the opaque limit as the modified Rosseland approximation is. However, as was pointed out earlier that the temperature gradient obtained from equation (22) is steeper near the boundary than that obtained from equation (19), the Rosseland approximation will yield a lower conduction heat flux and a higher radiation heat flux than the modified Rosseland approximation. Some results of the total heat flux calculated from equation (26) for $\theta_2 = 0{\cdot}5$, $\epsilon_1 = \epsilon_2 = 1$, and $\tau_0 = 10$ are presented in Fig. 1 which shows the improved accuracy of the results given by equation (26) over the results given by the Rosseland approximation.

For transparent wall equation (27) gives for $\theta_2 = 0{\cdot}5$, $\tau_0 = 10$, $\varepsilon_2 = 1$, $N = 1$, and $I_1^+ = 4$, i.e.

$$I_1 = 4\frac{\sigma}{\pi}T_1^4,$$

the heat flux,

$$q^+ = 0{\cdot}595.$$

While the Rosseland approximation gives the heat flux,

$$q^+ = 0{\cdot}325,$$

which would be of error 83 per cent if 0·595 is assumed to be reasonably accurate.

When radiation heat transfer becomes more

$$q^+ = \frac{3N(1-\theta_2) + (1-\theta_2^4)}{\dfrac{3\tau_0}{4} + \dfrac{\frac{3}{8} + \frac{1}{3}(1-\epsilon_1)}{N + \frac{2}{3}\epsilon_1} + \dfrac{\frac{3}{8} + \frac{1}{3}(1-\epsilon_2)}{(N/\theta_2^3) + \frac{2}{3}\epsilon_2}} \tag{26}$$

for opaque walls, and

$$q^+ = \frac{3N(1-\theta_2) + (1-\theta_2^4) + \left(\dfrac{1}{2} + \dfrac{\frac{3}{8}}{N+\frac{2}{3}}\right)(I_1^+ - 1)}{\dfrac{3\tau_0}{4} + \dfrac{\frac{3}{8}}{N + \frac{2}{3}} + \dfrac{\frac{3}{8} + \frac{1}{3}(1-\epsilon_2)}{(N/\theta_2^3) + \frac{2}{3}\epsilon_2}} \tag{27}$$

for opaque wall at boundary 2 and transparent wall with $I_1(\mu) = I_1 =$ constant at boundary 1.

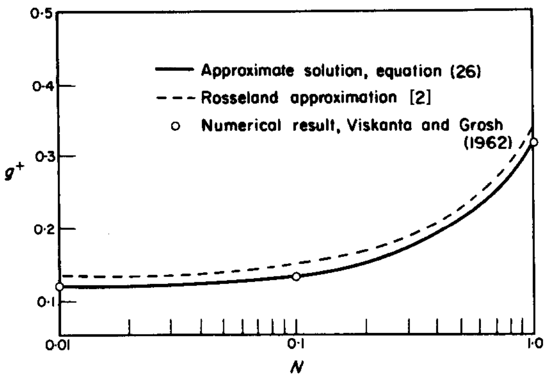

FIG. 1. Heat transfer for combined radiation and conduction between parallel black walls, $\theta_2 = 0{\cdot}5$ and $\tau_0 = 10$.

predominant, Taylor's expansion of the temperature function near the boundary becomes a poorer representation. Finally a singularity of the temperature function appears at the boundary when the radiation heat transfer becomes the sole mechanism of heat transfer. It seems that the singular perturbation technique has to be employed to solve this problem which will be discussed in Section 4.1.

3.2. *The transparent limit*

In the transparent limit, $J(\bar{x}) \cong J_0(\bar{x})$. The equation of conservation of energy becomes

$$\frac{N}{\tau_0^2}\,\mathrm{div}(\mathrm{grad}\,\theta) = \theta^4 - \frac{J_0}{(\sigma/\pi)T_1^4}. \tag{28}$$

Consider, for example, a plane layer again. The equation of conservation of energy then becomes [2]

$$N\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = \theta^4 - J_0^+, \tag{29}$$

where $J_0^+ \equiv J_0/(\sigma/\pi)\,T_1^4$. The boundary conditions of equation (29) are the temperatures at both boundaries

$$\theta(0) = 1 \qquad \text{and} \qquad \theta(\tau_0) = \theta_2, \tag{30}$$

and the heat fluxes at both boundaries in terms of incident intensities, I_1, and I_2,

$$q^+ = -4N\frac{\mathrm{d}\theta(0)}{\mathrm{d}\tau} + I_1^+ - I_2^+(1-2\tau_0) - 2\int_0^{\tau_0}\theta^4(\tau)\,\mathrm{d}\tau \tag{31a}$$

and

$$q^+ = -4N\frac{\mathrm{d}\theta(\tau_0)}{\mathrm{d}\tau} + I_1^+(1-2\tau_0) - I_2^+ + 2\int_0^{\tau_0}\theta^4(\tau)\,\mathrm{d}\tau. \tag{31b}$$

Now J_0^+ is a constant for a plane layer. Equation (29) can be readily integrated with respect to θ,

$$\frac{1}{2}N\left(\frac{\mathrm{d}\theta(0)}{\mathrm{d}\tau}\right)^2 - \frac{1}{2}N\left(\frac{\mathrm{d}\theta(\tau_0)}{\mathrm{d}\tau}\right)^2 = \tfrac{1}{5}(1-\theta_2^5) - J_0^+(1-\theta_2), \tag{32}$$

which satisfies boundary condition (30). Equation (29) can also be integrated with respect to τ,

$$N\frac{\mathrm{d}\theta(0)}{\mathrm{d}\tau} - N\frac{\mathrm{d}\theta(\tau_0)}{\mathrm{d}\tau} = \tau_0 J_0^+ - \int_0^{\tau_0}\theta^4\,\mathrm{d}\tau. \tag{33}$$

Dividing each side of equation (32) by the corresponding side of equation (33) and substituting the resulting equation into the sum of equations (31a) and (31b) result in

$$q^+ = (I_1^+ - I_2^+)(1-\tau_0) + \frac{4NJ_0^+(1-\theta_2) - [(4N/5)(1-\theta_2^5)]}{\tau_0 J_0^+ - \int_0^{\tau_0}\theta^4\,\mathrm{d}\tau}. \tag{34}$$

For

$$N/\tau_0 \gg 1, \qquad \mathrm{d}^2\theta/\mathrm{d}\xi^2 \ll \tau_0 \to 0$$

from equation (29). Then the temperature distribution $\theta(x)$ becomes very close to the temperature distribution for pure conduction, i.e.

$$\theta(\xi) = 1 - (1-\theta_2)\,\xi. \tag{35}$$

This gives

$$\int_0^{\tau_0}\theta^4\,\mathrm{d}\tau = \frac{\tau_0}{1-\theta_2}\,\frac{1-\theta_2^5}{5}.$$

Substituting this into equation (34), it follows

$$q^+ = (I_1^+ - I_2^+)(1 - \tau_0) + \frac{4N}{\tau_0}(1 - \theta_2)$$

$$\cong \frac{I_1^+ - I_2^+}{1 + \tau_0} + \frac{4N}{\tau_0}(1 - \theta_2). \qquad (36)$$

Therefore in the transparent limit and for $N/\tau_0 \gg 1$, the interaction between radiation and conduction is negligible and the total heat flux is equal to the superposition of the radiation heat flux in the absence of conduction and the conduction heat flux in the absence of radiation.

It seems also feasible that the temperature gradients at both boundaries be first obtained in terms of the integral, $\int_0^{\tau_0} \theta^4 \, d\tau$, by solving equations (32) and (33). Then one of them be substituted into equation (31a), for instance, to solve for the integral, $\int_0^{\tau_0} \theta^4 \, d\tau$, in terms of q^+. The total heat flux may then be determined from equation (31b) with the values of $d\theta(\tau_0)/d\tau$ and $\int_0^{\tau_0} \theta^4 \, d\tau$ thus given. In this way solutions with temperature distribution, $\theta(\xi)$, not subjected to equation (35) may be obtained. However as radiation becomes more and more predominant and the temperature distribution approaches that of the mean intensity, $(\sigma/\pi)T^4(x) \to J_0(x)$, the use of the above algebraic steps becomes ambiguous, because the temperature function has singularities at boundaries in the absence of conduction, and the temperature gradients at boundaries are undefined.

In the radiation predominant, transparent case the singular perturbation technique can still be employed for the solution of equation (29). Since the application of this technique requires no restriction of the magnitude of opacity, it will be discussed in Section 4.1 for the general case.

4. COMBINED RADIATION AND CONDUCTION—PART B

Encouraged by the usefulness of considering the opaque and transparent limits of a medium in radiative equilibrium, the solution to the problem of combined radiation and conduction was also sought in these two limits in the last section. It was found, however, that in the radiation predominant case the solution to the problem calls for the use of the singular perturbation technique regardless of whether the medium is opaque or transparent. Therefore the methods in Section 3 fail to yield a pair of asymptotic solutions which are complementary with each other.

Since the radiation predominant case presents itself naturally as the case whose solution calls for the application of the singular perturbation technique, it will be discussed first in Section 4.1. Then its complementary case, the conduction predominant case, will be discussed in Section 4.2.

4.1 *Radiation predominant case*

One of the purposes of this section is to indicate that the same calculation procedure used in [5] can be applied to a larger class of body-geometries other than plane-layer.

Substitution of equation (2) into the equation of conservation of energy results in

$$-\operatorname{div} \vec{q}_c = 4\pi\rho\alpha(S - J). \qquad (37)$$

Since there exist temperature slips at boundaries for a medium in radiative equilibrium, the effect of conduction appears first in a thin layer near the boundaries. It is sufficient to consider such a thin layer to be a plane layer. In this boundary layer, equation (37) reduces into the dimensionless form,

$$\theta^4 - N\frac{d^2\theta}{d\tau^2} = J^+ \cong J_r^+, \qquad (38)$$

where the dimensionless mean intensity, J^+, is approximated by that of the medium in radiative equilibrium, J_r^+, as a first approximation.

Equation (38) is identical to equation (33) in [5]. Hence the method of solution in [5] can be immediately applied to solve equation (38). In obtaining equation (38), however, no restriction to plane-layer geometry is made and no differential approximation is used. The only approximation made, $J \cong J_r$, is equivalent to the physical assumption that the effect of conduction is so small that it changes negligibly the mean intensity of the medium from that in radiative equilibrium. The temperature profile, of course, will be quite different from that in radiative equilibrium where

$$\theta^4 = J_r^+ \text{ everywhere.}$$

4.2 *Conduction predominant case*

For the conduction predominant case the perturbation of radiation on conduction does not limit its effect in thin boundary layers. Hence equation (37) cannot be solved separately in distinct geometric regions. But in this case, the differential operator, $d^2/d\tau^2$, can be treated by using Green's function because the temperature function and its second order derivative are functions whose square is integrable [10].

For simplicity, consider a plane-layer with black opaque walls. The equation of conservation of energy in dimensionless form is

$$\frac{N}{\tau_0^2}\frac{d^2\theta}{d\xi^2} = -\tfrac{1}{2}E_2(\tau_0\xi) - \tfrac{1}{2}\theta_2^4 E_2(\tau_0 - \tau_0\xi) + \theta^4(\xi) - \tfrac{1}{2}\tau_0\int_0^1 \theta^4(t)E_1(\tau_0|\xi - t|)\,dt. \quad (39)$$

The Green's function for differential operator, $-d^2/d\xi^2$, with Dirichlet boundary conditions is equal to

$$g(\xi, t) = \xi(1 - t)H(t - \xi) + t(1 - \xi)H(\xi - t), \quad (40)$$

where $H(t)$ is the unit step function. A nonlinear integral equation is obtained by using the Green's function,

$$\frac{N}{\tau_0^2}\theta = f(\xi) + \int_0^1 \theta^4(t)\left[-g(\xi, t) + \tfrac{1}{2}\tau_0 h(\xi, t)\right]dt, \quad (41)$$

where $f(\xi)$ and $h(\xi, t)$ are known functions and are given as follows:

$$f(\xi) = \frac{N}{\tau_0^2}[1 - (1 - \theta_2)\xi] + \int_0^1 dt\, g(\xi, t)\left[\tfrac{1}{2}E_2(\tau_0 t) + \tfrac{1}{2}\theta_2^4 E_2(\tau_0 - \tau_0 t)\right] \quad (42)$$

and

$$\begin{aligned} h(\xi, t) &= \int_0^1 g(\xi, s)E_1(\tau_0|s - t|)\,ds \\ &= H(t - \xi)\left[\frac{2}{\tau_0}\xi(1 - t) - \frac{1}{\tau_0^2}E_3(\tau_0 t - \tau_0\xi) + \frac{1}{\tau_0^2}(1 - \xi)E_3(\tau_0 t) + \frac{1}{\tau_0^2}\xi E_3(\tau_0 - \tau_0 t)\right] \\ &\quad + H(\xi - t)\left[\frac{2}{\tau_0}t(1 - \xi) - \frac{1}{\tau_0^2}E_3(\tau_0\xi - \tau_0 t) + \frac{1}{\tau_0^2}(1 - \xi)E_3(\tau_0 t) + \frac{1}{\tau_0^2}\xi E_3(\tau_0 - \tau_0 t)\right] \end{aligned} \quad (43)$$

If the pure conduction solution,

$$\theta(\xi) = 1 - (1 - \theta_2)\xi,$$

is substituted into the integrand of equation (41), the integration can be readily carried out to give the first order term in a regular perturbation series. The resulting algebraic expression is so lengthy that it would not be presented here. An example of this result is given in Fig. 2.

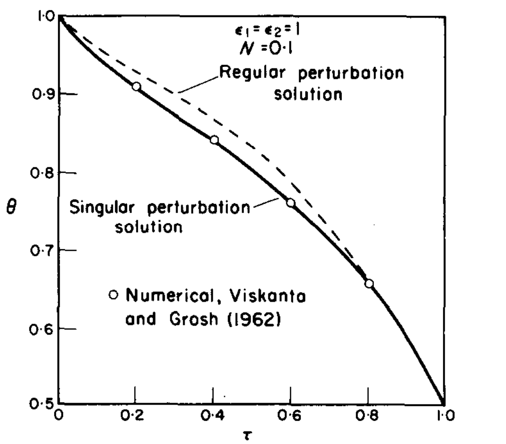

FIG. 2. Temperature profile for combined radiation and conduction between parallel black walls, $\theta_2 = 0{\cdot}5$, $\tau_0 = 1$ and $N = 0{\cdot}1$.

Also presented in Fig. 2 are the results from singular perturbation solution based on the calculation in [11] and the numerical result of Viskanta and Grosh [12]. The value of parameter, N, is chosen to be $N = 0{\cdot}1$. Recall the definition of N,

$$N \equiv \frac{k\rho\alpha}{4\sigma T_1^3}.$$

Hence $N = 0{\cdot}1$ implies radiation is predominant at wall 1. While at wall 2 $N = 0{\cdot}1$ implies

$$N' \equiv \frac{k\rho\alpha}{4\sigma T_2^3} = N\left(\frac{T_1}{T_2}\right)^3 = 0{\cdot}8 = O(1),$$

that is, radiation and conduction are equally important. This leads to the expectation that the singular perturbation solution will agree with numerical result near wall 1 while the regular perturbation solution will not. This expectation is confirmed in Fig. 2. It is found, moreover, that both perturbation solutions agree with numerical results near wall 2 where radiation and conduction are equally important.

5. DISCUSSION AND CONCLUSION

It should be clear from the discussion on combined radiation and conduction in Section 3 that there exist two distinct boundary conditions —the radiation boundary condition and the conduction boundary condition. Any consistent approximation method should include both boundary conditions when both radiation and conduction are important.

It should also be clear from Section 3 that the opaque limit solution and the transparent limit solution in their present forms fail to constitute a complementary solution for the combined radiation and conduction problems. On the other hand it was shown in Section 4 that the radiation predominant solution and the conduction predominant solution constitute a complementary solution. This conclusion is illustrated in Fig. 3. The solid curve which separates the radiation predominant region and the conduction predominant region is plotted by equalizing the following parameter to be unity

$$C(N, \tau_0) = \begin{cases} N/\tau_0^2, & \tau_0 \ll 1. \quad (44a) \\ N \quad , & \tau_0 \gg 1. \quad (44b) \end{cases}$$

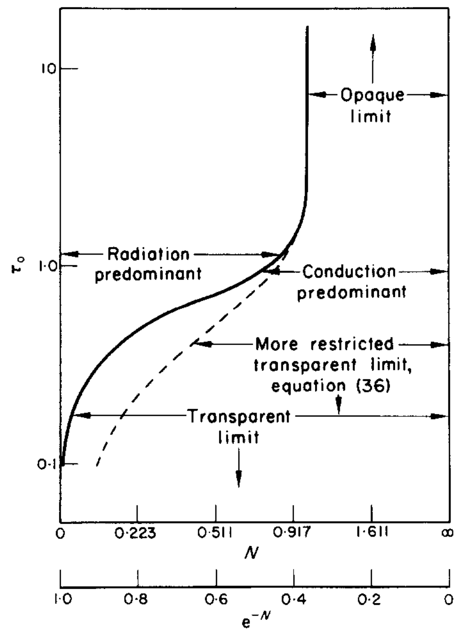

FIG. 3. Regions of various limits for combined radiation and conduction between parallel walls.

The parameter C characterizes the relative importance of conduction and radiation on the determination of temperature profile and is generally a function of wall emissivities and the ratio of wall temperatures also. Here the attention is directed only to the two most important parameters C depends on, i.e. N and τ_0. The asymptotic dependence of C on N and τ_0 may be obtained from the consideration in Section 3.2 and Section 3.1.

Not only the attempt to consider only the opaque limit and the transparent limit is unsatisfactory, it is, in some respects, even undesirable. This is referring to the possibility that the differential approximate formulation [14, 15] would have been easily obtained by combining equations (1) and (2) if not for the fact that the "Rosseland approximation" had always been identified with the more restricted approximation, equation (19), in the opaque limit. Rosseland [6] originally presented equation (1) not equation (19) as an approximate formula for the radiation heat flux, then went on to show that the condition from which equation (1) was derived is fulfilled in the opaque limit. It was noted [11] later that this condition is also fulfilled for transparent medium ($\tau_0 = 0$) in a plane-layer. In fact a more careful examination of equation (1) reveals that at $\tau_0 = 0$, J is mathematically forced to become uniform throughout the whole medium. Therefore, solution of equations (1) and (2) will be valid at $\tau_0 = 0$ only for systems whose mean intensities are truly uniform at $\tau_0 = 0$. Regardless of its limitation the differential approximation, equation (1), is much more useful than the "Rosseland approximation", equation (19), which is just a particular form of equation (1) in the opaque limit.

Another point is worth emphasizing here. The method of integral equations [e.g. equation (41)] is often accused of lacking effectiveness. This reproach is especially justified with regard to multi-dimensional problems. In this respect, the differential approximation, equation (1), appears to be very useful in its proper content.

REFERENCES

1. J. O. HIRSCHFELDER, C. F. CURTISS and R. B. BIRD, *Molecular Theory of Gases and Liquids*, pp. 720–727. John Wiley, New York (1954).
2. R. D. CESS, The interaction of thermal radiation with conduction and convection heat transfer, in *Advances in Heat Transfer*, edited by T. F. IRVINE JR. and J. P. HARTNETT, Vol. 1, pp. 1–50. Academic Press, New York (1964).
3. H. C. HOTTEL, Some problems in radiative transport, Lecture presented at Int. Heat Trans. Confer., Boulder, Colorado (1961).
4. D. B. OLFE and S. S. PENNER, Some comments on radiation slip, *Jl Quantve Spectros. & Radiat. Transf.* **4**, 229 (1964).
5. L. S. WANG and C. L. TIEN, Study of the interaction between radiation and conduction by a differential method, in *Proceedings of the Third International Heat Transfer Conference*, Vol. 5, 190–199. Am. Inst. Chem. Engrs, New York (1966).
6. S. ROSSELAND, *Astrophysik auf Atom-Theoretischer Grundlage*. Springer Verlag, Berlin (1931).
7. M. A. HEASLET and R. F. WARMING, Radiative transport and wall temperature slip in an absorbing planar medium, *Int. J. Heat Mass Transfer* **8**, 979 (1965).
8. C. L. TIEN and L. S. WANG, Band absorption laws, gas body geometries and the mean beam length, in *Proceedings of the 1965 Heat Transfer Fluid Mechanics Institute*, pp. 345–357. Stanford University Press, Stanford (1965).
9. J. WALLACE, The opaque limit in radiation gas dynamics, AIAA Paper No. 65–70 (1965).
10. B. FRIEDMAN, *Principles and Techniques of Applied Mathematics*. John Wiley, New York (1956).
11. L. S. WANG, Differential methods for combined radiation and conduction, Ph.D. thesis in engineering, Univ. of California, Berkeley, California (1965).
12. R. VISKANTA and R. J. GROSH, Heat transfer by simultaneous conduction and radiation in an absorbing medium, *J. Heat Transfer* **84**, 63 (1962).
13. R. VISKANTA and R. J. GROSH, Effect of surface emissivity on heat transfer by simultaneous conduction and radiation in an absorbing medium, *Int. J. Heat Mass Transfer* **5**, 729 (1962).
14. S. C. TRAUGOTT, A differential approximation for radiative transfer with application to normal shock structure, in *Proceedings of the 1963 Heat Transfer and Fluid Mechanics Institute*, 1–13. Stanford University Press, Stanford (1963).
15. P. CHENG, Dynamics of a radiating gas with application to flow over a wavy wall, *AIAA Jl* **4**, 238 (1966).

Résumé—On présente une étude de deux classes de théories limites pour les problèmes de transport de chaleur par rayonnement: d'une part, le cas limite opaque et le cas limite transparent et d'autre part le cas limite avec prédominance du rayonnement et le cas limite avec prédominance de la conduction.

Dans la première situation, il n'y a pas de traitement valable uniformément sans s'occuper si le rayonnement ou la conduction sont prédominants. Dans la seconde situation, cependant, on a obtenu un traitement uniformément valable quelle que soit la valeur de l'opacité. En conséquence, il offre un moyen plus logique que dans la première situation pour résoudre des problèmes de rayonnement et de conduction combinés. La schéma comprend la résolution d'une équation différentielle non-linéaire dans le cas avec prédominance du rayonnement et une équation intégrale non-linéaire dans le cas avec prédominance de la conduction.

Zusammenfassung—Für die beiden Arten der Grenzbetrachtung beim Wärmetransport durch Strahlung wird hier eine Untersuchung durchgeführt: Für die Betrachtung einer undurchlässigen Begrenzung und einer durchlässigen einerseits und für jene der strahlungsdominierenden und der leitungsdominierenden Begrenzung andererseits. Für den ersten Fall ist eine einheitliche Behandlung, ungeachtet der Strahlungs- oder Leitungsdominanz nicht zu erreichen; im zweiten Fall jedoch ist diese möglich, ungeachtet des Wertes der Durchlässigkeit. Damit bietet sich ein sinnfälligerer Lösungsweg für kombinierte Strahlungs- und Leitungsprobleme als im ersten Fall. Das Schema umfasst die Lösung einer nichtlinearen Differentialgleichung im strahlungsdominierenden Fall und eine nichtlineare Integralgleichung im leitungsdominierenden Fall.

Аннотация—Проведен анализ двух видов предельных задач лучистого теплообмена: предельной непрозрачности или предельной прозрачности, с одной стороны, и преобладания лучистого или кондуктивного обмена с другой стороны. Для первого случая не существует метода, равноценного для преобладающей радиации и преобладающей теплопроводности. Во втором случае получен метод, применимый для любой степени непрозрачности. Поэтому он дает более логичный способ решения задач совместной радиации и теплопроводности. Методика включает решение нелинейного дифференциального уравнения в случае преобладающей радиации и решение нелинейного интегрального уравнения в случае преобладающей теплопроводности.